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Stability of an Orbiting Ring

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The stability of a ring in a circular orbit is investigated under two different assumptions: 1) the ring is a perfectly flexible but inextensible chain; 2) the chain is perfectly flexible but has longitudinal elasticity. Under the first assumption, the first mode is always unstable. Under the second assumption, this mode can be stable if the spring constant is sufficiently weak, but numerical studies indicated that some higher mode was always unstable. Moreover, any elastic damping in the chain destabilizes the first mode. The ring can, however, be stabilized by local length adjustments using feedback.

Introduction

A FEW decades from now there will be enough satellites in synchronous orbit to cause a serious traffic problem. It is tempting to consider the eventual possibility of cabling together a very large number of satellites to form a ring encircling the Earth at geosynchronous altitude.[1] Moreover, if the satellite altitude is very slightly increased, synchronous rate is still maintainable with the cables in tension. This solution to the traffic problem is attractive provided that such an orbiting ring is not passively unstable.† The stability of such a ring is the topic of this paper.

The opinion that the future colonization of space will depend to a large degree on a gigantic structure attached to the Earth but extending into space is expressed by Clarke in Ref. 2. This structure would consist of several "orbital towers"[3] joined by a ring. The construction of the towers, which reach down from synchronous altitudes to the Earth's surface and reach up an even greater distance so that the total centrifugal force on a tower slightly exceeds the total gravity force, might be accomplished by deploying them from orbit. However, the possibility of constructing the ring before the towers depends again on the stability of a ring.

The stability of an orbiting ring will be examined by treating the ring as a continuous, perfectly flexible chain of uniform density in circular orbit at any altitude about a planet whose oblateness is ignored. Longitudinal elasticity of the chain will be included.

Analysis

If the nominal orbit has radius R and angular rate n, the equilibrium of an element of the ring of angular length $\mathrm{d}\alpha$ requires that

$$T_0 \mathrm{d}\alpha = (\sigma R \mathrm{d}\alpha)\,[n^2 R - (\mu/R^2)]$$

where T_0 is the equilibrium tension, σ the density per unit length, and μ/R^2 the gravitational acceleration. For positive tension T_0, we clearly need

$$n^2 > \mu/R^3$$

Let the point α of the ring be displaced in the nominal orbital plane from $r=R$, $\theta=\alpha+nt$ to $r=R+x(\alpha,t)$, $\theta=\alpha+nt+(y(\alpha,t)/R)$, and out of the nominal plane by a distance $z(\alpha,t)$, where $x(\alpha,t)$, $y(\alpha,t)$, $z(\alpha,t)$ and their partial derivatives $w{\cdot}r{\cdot}t{\cdot}\alpha$ are small compared to R. The displaced element of arc is

$$\mathrm{d}s = \{(x_\alpha \mathrm{d}\alpha)^2 + (z_\alpha \mathrm{d}\alpha)^2 + [(R+x)(1+y_\alpha/R)\mathrm{d}\alpha]^2\}^{1/2}$$
$$= (R+x+y_\alpha)\mathrm{d}\alpha$$

neglecting second-order terms in the small quantities x, etc. The increase in tension T may be assumed to be proportional to the extension per unit length, i.e.,

$$T = T_0 + k(x+y_\alpha)$$

where k is a constant.

The angle from the undisplaced to the displaced tangents is $\phi(\alpha,t) = y/R - (1/R)x_\alpha$, measured in the same sense as α and the orbital motion. The net force on the displaced ring element has the following components:

Along the undisplaced radius

$$F_x = 2\mu(\sigma R \mathrm{d}\alpha)x/R^3 - T_0\phi_\alpha \mathrm{d}\alpha - k(x+y_\alpha)\mathrm{d}\alpha$$

Along the undisplaced tangent

$$F_y = -\mu(\sigma R\mathrm{d}\alpha)y/R^3 - \phi(T_0\mathrm{d}\alpha) + \frac{\partial T}{\partial \alpha}\mathrm{d}\alpha$$

Out-of-plane

$$F_z = -\mu(\sigma R \mathrm{d}\alpha)z/R^3 + T_0 z_{\alpha\alpha}\mathrm{d}\alpha$$

The equations of motion of the ring are thus:

$$x_{tt} - 2ny_t - n^2 x = \frac{2\mu}{R^3}x - \left(n^2R - \frac{\mu}{R^2}\right)\left(\frac{y_\alpha}{R} - \frac{x_{\alpha\alpha}}{R}\right) - \frac{k}{\sigma R}(x+y_\alpha)$$

$$y_{tt} + 2nx_t - n^2 y = -\frac{\mu}{R^3}y - \left(n^2R - \frac{\mu}{R^2}\right)\left(\frac{y}{R} - \frac{x_\alpha}{R}\right) + \frac{k}{\sigma R}(x_\alpha + y_{\alpha\alpha})$$

$$z_{tt} = -\frac{\mu}{R^3}z + \left(n^2 R - \frac{\mu}{R^2}\right)\frac{z_{\alpha\alpha}}{R}$$

Introducing the nondimensional parameters

$$\omega^2 = \frac{T_0}{n^2\sigma R^2} = 1 - \frac{\mu}{n^2R^3} < 1 \qquad \omega_1^2 = \frac{k}{n^2 \sigma R}$$

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†This question was posed to the author by S. Bowen of the NASA Ames Research Center.

and the nondimensional time $\tau = nt$, the equations of motion become

$$x_{\tau\tau} - 2y_\tau - (3 - 2\omega^2)x - \omega^2(x_{\alpha\alpha} - y_\alpha) + \omega_I^2(x + y_\alpha) = 0$$

$$y_{\tau\tau} + 2x_\tau - \omega^2 x_\alpha - \omega_I^2(x_\alpha + y_{\alpha\alpha}) = 0$$

$$z_{\tau\tau} + (1 - \omega^2)z - \omega^2 z_{\alpha\alpha} = 0$$

We look next for a modal solution with x,y,z proportional to $e^{j(m\alpha + \Omega_m \tau)}$, where m is an integer (clearly x,y,z must have period 2π in α). The coupled x,y equations lead to the following characteristic equation:

$$\Omega_m^4 - [1 + (m^2+2)\omega^2 + (m^2+1)\omega_I^2]\Omega_m^2 + 4m(\omega^2 + \omega_I^2)\Omega_m$$
$$+ m^2[m^2\omega^2\omega_I^2 - \omega^4 - 3\omega_I^2] = 0$$

The z equation leads to

$$\Omega_m^2 = m^2\omega^2 + (1 - \omega^2)$$

The ring is unstable unless each characteristic equation leads to real, distinct values for Ω_m. The out-of-plane motion is clearly linearly stable. We notice, as a check, that it includes a mode $z(\alpha,t) = \exp[j(\alpha + \tau + \text{const})]$, corresponding to a small constant change in the orbital plane.

A check on the in-plane equations is provided by observing that they are satisfied, for the special case $\mu = 0$ (no gravity) by,

$$x = -y_\alpha \qquad y = \text{ any } f(\alpha + nt)$$

corresponding to a constant in-plane distortion of the chain without change of length and hence tension. This possible behavior of a flexible chain, moving along itself at constant speed in the absence of external forces, is well-known.

The Inextensible Ring

The stability of the inextensible ring is easily described: Here $\omega_I^2 = \infty$ and the in-plane characteristic equation is

$$(m^2+1)\Omega_m^2 - 4m\Omega_m - m^2(m^2\omega^2 - 3) = 0$$

For sufficiently large m, the last term is negative, provided the tension is positive and linear stability is assured. However for $m = 1$ we obtain $\Omega_I = 1 \pm j\sqrt{(1-\omega^2)}/2$ and this mode is unstable. Here x and y are proportional to $\exp[j(\alpha + nt) \pm t\sqrt{\mu/(2R^3)}]$ which implies a rigid‡ translation (Fig. 1) of the whole ring away from a centered position due to the gravity field. Indeed, a small rigid translation $\vec{\xi} = \xi \hat{i}_0$ off center produces a gravitational force

$$\vec{F} = \frac{\mu}{R^3}\int_0^{2\pi}(3\hat{r}(\alpha)\hat{r}^T(\alpha) - I)\vec{\xi}(R\sigma d\alpha)$$

in which $\hat{r}(\alpha) = \hat{i}_0\cos\alpha + \hat{j}_0\sin\alpha$. This easily simplifies to

$$\vec{F} = \tfrac{1}{2}(\mu/R^3)\vec{\xi}(2\pi R\sigma)$$

and hence

$$\ddot{\vec{\xi}} = \tfrac{1}{2}(\mu/R^3)\vec{\xi}$$

The Extensible Ring

The in-plane characteristic equation has been investigated numerically for various values of m, ω^2, and ω_I^2. It was found that the first mode, $m = 1$, could become linearly stable if ω_I^2 were sufficiently small in comparison to ω^2, corresponding to $k < T_0/2\pi R$, which is not possible, incidentally, for a chain of strictly linear characteristics. The hoped-for stability, however, was never found; stability for $m = 1$ was always accompanied by instability of some higher mode. Stability is indeed impossible for $m = 1$ when elastic damping is taken into account.

To see this, we replace k by $k(1 + \beta j\Omega)$, where β is real and positive and hence ω_I^2 by $j\beta K[\Omega - (j/\beta)]$, where K is the previous ω_I^2, a positive real number. A typical root-locus for Ω_I vs K with fairly small β and ω^2 is shown in Fig. 2. There is a branch entering the lower Ω_I half-plane and instability results for all $K > 0$. For larger β, the upper half-plane pattern is changed, but the lower half-plane is not essentially different. For larger ω^2, the locus starts from a pair of real poles together with a pair of complex conjugate poles. In this case, one branch lies entirely in the lower half-plane. In all cases then, a positive damping coefficient β implies instability at $m = 1$.

A Stabilizing Control Scheme

We turn now to the possibility of partially controlling the ring by local adjustments of the cable lengths. The control is, at best, partial, since an incorrect total angular momentum cannot be corrected.

In our continuous model, the local length adjustment is accounted for by replacing the original tension formula by

$$T = T_0 + k(x + y_\alpha - u)$$

where $u(\alpha,t)$ is the local length increase. The first two dimensionless equations of motion now have driving terms $\omega_I^2 u$ and $-\omega_I^2 u_\alpha$, respectively.

If we assume a control law

$$u = \phi(j\Omega)x + \psi(j\Omega)y$$

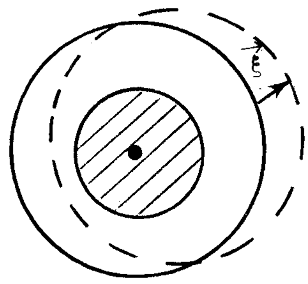

Fig. 1 Rigid translation.

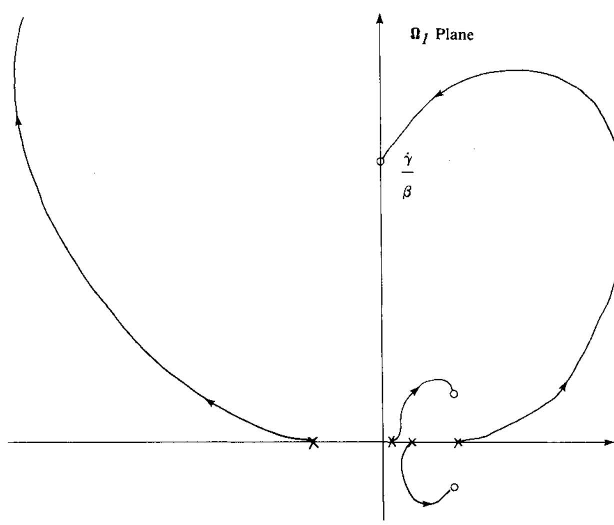

Fig. 2 Root locus for Ω_I vs $K = \text{Re}(\omega_I^2)$, fixed β.

‡The instability of an orbiting rigid ring was recognized by Laplace.

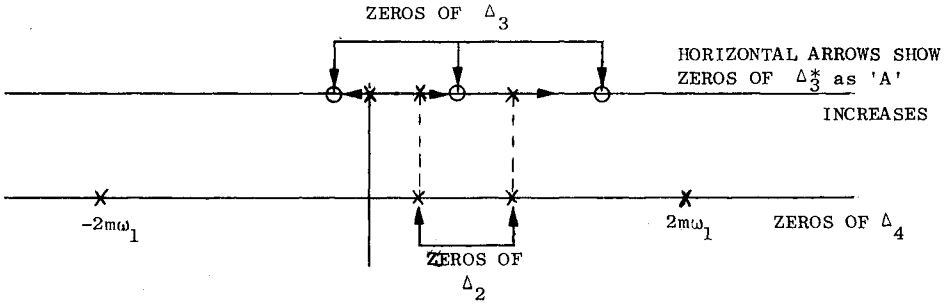

Fig. 3 Zeros of Δ_3^* and Δ_4.

the characteristic equation for Ω_m now has additional additive terms:

$$+\omega_1^2\phi(j\Omega_m)\,[\Omega_m^2-2m\Omega_m+m^2\omega^2]$$

$$-j\omega_1^2\psi(j\Omega_m)\,[m\Omega_m^2-2\Omega_m+m(3-\omega^2+m^2\omega^2)]$$

If, in particular, we choose $\psi\equiv 0$ and

$$\phi(j\Omega)=1-(3-2\omega^2)/\omega_1^2-3m^2$$

that is,

$$u=[1-(3-2\omega^2)/\omega_1^2]x+3x_{\alpha\alpha}$$

the characteristic equation for Ω_m becomes

$$\Delta_4(\Omega_m)\equiv\Omega_m^4-(4+m^2\omega^2)\Omega_m^2+6m\Omega_m-m^2\omega^2(3-\omega^2)$$

$$-\omega_1^2\{4m^2\Omega_m^2-2m(1+3m^2)\Omega_m+m^2(3-\omega^2+2m^2\omega^2)\}=0$$

If we now assume that ω_1^2 is real $\gg 1$, thus ignoring for the moment any elastic damping, and that $\omega^2\ll 1$, the zeros of $\Delta_4(\Omega_m)$ are closely approximated by $\Omega_m^2-4m^2\omega_1^2=0$ and $\Delta_2(\Omega_m)\equiv 4m^2\Omega_m^2-2m(1+3m^2)\Omega_m+m^2(3+2m^2\omega^2)=0$ leading to four real characteristic roots. (The discriminant of the second quadratic is found to be $4m^2\{9(m^2-1)^2+4(3m^2-2)\}$, which is positive for all $m\geq 1$.)

Note that for the mode $m=1$, our control law is essentially $u=-2x$, implying a tightening of the cable lengths on the further part of the rigidly displaced ring, and a loosening on the near part as one would expect.

It remains to introduce damping by means of additional feedback. We shall continue to suppose that $\omega_1^2\gg 1$ and $\omega^2\ll 1$. Still ignoring the elastic damping parameter α, suppose that

$$u=\left(1-\frac{3-2\omega^2}{\omega_1^2}\right)x+3x_{\alpha\alpha}-\frac{A}{\omega_1^2}\left(x_\tau-\frac{1}{4}\omega^2x_{\tau\alpha\alpha}\right)$$

$$+\frac{A}{\omega_1^2}\left(\frac{1}{4}y+\frac{1}{12}\omega^2y_{\alpha\alpha}\right)$$

where A is real and positive. The "uniform" mode $m=0$ leads to

$$x_{\tau\tau}+Ax_\tau-2y_\tau-(A/4)y=0\qquad 2x_\tau+y_{\tau\tau}=0$$

with characteristic equation, writing s for $j\Omega_0$:

$$s^2(s^2+4)+As(s^2+\tfrac{1}{2})=0$$

leading to the damping out of all quantities except x. Indeed, an initial error in total angular momentum leads to $2x+y_\tau\neq 0$, but our control law removes the angular position error y by adjusting the cable length and hence the radial increment x.

The higher modes $(m\geq 1)$ now have characteristic equation:

$$\Delta_4(\Omega_m)-jA\Delta_3(\Omega_m)=0$$

where

$$\Delta_3(\Omega_m)\equiv\left(1+\frac{1}{4}m^2\omega^2\right)\Omega_m^3-m\left(\frac{7}{4}+\frac{7}{12}m^2\omega^2\right)\Omega_m^2$$

$$+\left(\frac{1}{4}m^4\omega^4+\frac{7}{6}m^2\omega^2-\frac{1}{2}\right)\Omega_m+\frac{m}{12}(3-m^2\omega^2)^2$$

(Here we have dropped terms in ω^2 without coefficient m^2.) The zeros of $\Delta_3(\Omega_m)$ are not only real for all m but separate those of $\Delta_4(\Omega_m)$, as will be shown in the Appendix. The root-locus for Ω_m vs A thus has all four branches entering the upper-half Ω_m plane as soon as $A>0$. Our control law thus stabilizes all modes, at least in the absence of an elastic damping parameter β.

Lastly, we include the elastic damping parameter β, so that $\omega_1^2=K(1+\beta j\Omega)$. Our chosen control law is

$$u=\left[1-\frac{3-2\omega^2}{K(1+\beta s)}\right]x+3x_{\alpha\alpha}-\frac{As}{K(1+\beta s)}\left(x-\frac{1}{4}\omega^2x_{\alpha\alpha}\right)$$

$$+\frac{A}{K(1+\beta s)}\left(\frac{1}{4}y+\frac{1}{12}\omega^2y_{\alpha\alpha}\right)$$

The resulting characteristic equation is

$$\Delta_4(\Omega_m)-j\{A\Delta_3(\Omega_m)+K\beta\Omega_m\Delta_2(\Omega_m)\}=0$$

Moreover, the zeros of $\Delta_3^*(\Omega_m)\equiv A\Delta_3(\Omega_m)+K\beta\Omega_m\Delta_2(\Omega_m)$, like those of $\Delta_3(\Omega_m)$, separate the zeros of $\Delta_4(\Omega_m)$ (see Fig. 3). Satisfactory damping requires $A\omega^2$ of the same order of magnitude as $K\beta$.

Our control law tacitly assumes knowledge of the system parameters K,β,ω. Small error factors in this knowledge should have little effect. However, the angular position error y in the uniform mode $(m=0)$ will not be completely removed. This is easily remedied by adding a small "integral control" term $\Delta u=\gamma y/Ks$, where $\gamma\ll 1$.

Appendix

A proof is presented here that, for all $m\geq 1$, the zeros of $\Delta_2(\Omega)$ separate those of $\Delta_3(\Omega)$, and hence the zeros of $\Delta_3(\Omega)$ separate those of $\Delta_4(\Omega)$.

If $\Delta_2(\Omega)=0$,

$$\Omega^2=[3m/2+1/(2m)]\Omega-(3+2\lambda)/4$$

where $\lambda=m^2\omega^2>0$, so that

$$\Delta_3(\Omega)\equiv\left(1+\frac{\lambda}{4}\right)\left[\left(\frac{3m}{2}+\frac{1}{2m}\right)\Omega^2-\frac{3+2\lambda}{4}\Omega\right]-m\left(\frac{7}{4}+\frac{7\lambda}{12}\right)\Omega^2$$

$$+\left(\frac{\lambda^2}{4}+\frac{7\lambda}{6}-\frac{1}{2}\right)\Omega+\frac{m}{12}(3-\lambda)^2\quad[\text{modulo }\Delta_2(\Omega)]$$

$$\equiv\left[\frac{1}{m}\left(\frac{1}{2}+\frac{\lambda}{8}\right)-m\left(\frac{1}{4}+\frac{5\lambda}{24}\right)\right]\Omega^2+\left(\frac{\lambda^2}{8}+\frac{23\lambda}{48}-\frac{5}{4}\right)\Omega$$

$$+\frac{m}{12}(3-\lambda)^2 \qquad [\text{modulo } \Delta_2(\Omega)]$$

$$\equiv m\left(\frac{15}{16}-\frac{7\lambda}{32}+\frac{3\lambda^2}{16}\right)-\frac{(4+\lambda)(3+2\lambda)}{32m}$$

$$+\Omega\left[\frac{1}{m^2}\left(\frac{1}{4}+\frac{\lambda}{16}\right)-m^2\left(\frac{3}{8}+\frac{5\lambda}{16}\right)-\frac{5}{8}+\frac{9\lambda}{16}+\frac{\lambda^2}{8}\right]$$

$$[\text{modulo } \Delta_2(\Omega)]$$

which vanishes at, say, $\Omega=\Omega^*$.

Suppose first that $\lambda \ll 1$ (m not too large), then

$$\Delta_3(\Omega)\cong\frac{15m}{16}-\frac{3}{8m}-\Omega\left(\frac{3m^2}{8}-\frac{1}{4m^2}+\frac{5}{8}\right)$$

$$[\text{modulo } \Delta_2(\Omega)]$$

and

$$\Omega^*\cong\frac{1}{2m}\,\frac{15m^2-6}{3m^2+5-2/m^2}$$

so that

$$\Delta_2(\Omega^*)\cong\frac{-9(12m^6-11m^4+2m^2)}{(3m^2+5-2/m^2)^2}<0$$

It follows that Ω^* lies between the two (real) zeros of $\Delta_2(\Omega)$ and hence that $\Delta_3(\Omega)$ is negative at the larger zero of $\Delta_2(\Omega)$ and positive at the smaller zero. But $\Delta_3(+\infty)=+\infty$ and $\Delta_3(-\infty)=-\infty$, and $\Delta_3(\Omega)$ evidently here has three real roots separated by those of $\Delta_2(\Omega)$.

Suppose next that $\lambda \geqq 0(1)$, so that $m \gg 1$. Then

$$\Delta_3(\Omega)\cong m\left(\frac{15}{16}-\frac{7\lambda}{32}+\frac{3\lambda^2}{16}\right)-m^2\Omega\left(\frac{3}{8}+\frac{5\lambda}{16}\right)$$

$$[\text{modulo } \Delta_2(\Omega)]$$

and

$$\Omega^*\cong\frac{30-7\lambda+6\lambda^2}{2m(6+5\lambda)}$$

so that

$$\Delta_2(\Omega^*)\cong\frac{-8m^3(\lambda-3)^2}{6+5\lambda}$$

Except for the unlikely case in which $\lambda-3$ is exactly zero for some integer m, $\Delta_2(\Omega^*)<0$ and the argument given for the case $\lambda \ll 1$ again applies.

The reader may reintroduce neglected terms with ω^2 and $1/\omega_1^2$, and verify that if $\sqrt{3}/\omega=m_1+\delta$, where $|\delta|\leq 1/2$, the middle zero of $\Delta_3(\Omega)$ exceeds the smaller zero of $\Delta_2(\Omega)$ by a positive definite quadratic form in ω and δ, while the corresponding zero of $\Delta_4(\Omega)$ differs from that of $\Delta_2(\Omega)$ by a much smaller quantity. The separation of the zeros of $\Delta_4(\Omega)$ by those of $\Delta_3(\Omega)$ still holds.

A "matching" region between the regions $\lambda \ll 1$ and $\lambda \geqq 0(1)$ has λ small but m large and both analyses apply.

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